

Regular submodules of torsion modules over a discrete valuation domain

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Abstract

A submodule W of a p -primary module M of bounded order is known to be regular if W and M have simultaneous bases. In this paper we derive necessary and sufficient conditions for regularity of a submodule.

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1 Introduction

Let R be a discrete valuation domain with maximal ideal Rp , and let M be a torsion module over R and W be a submodule of M . The submodule W is called *regular* [5, p.65], [6, p.102] if

$$p^n W \cap p^{n+r} M = p^n (W \cap p^r M) \quad (1.1)$$

holds for all $n \geq 0, r \geq 0$. The regularity condition (1.1) was introduced by Vilenkin [6] in his study of decompositions of topological p -groups. Kaplanski [5] showed that for a module M of bounded order (1.1) is necessary and sufficient for the existence of simultaneous bases of W and M . In this paper we shall identify two conditions which are equivalent to (1.1). One is related to a theorem of Baer [4, p.4] on the decomposition of elements in abelian p -groups, the other one was introduced by Ferrer, F. Puerta and X. Puerta [2] to characterize marked invariant subspaces of a linear operator.

Notation and definitions: The letters $\mathcal{U}, \mathcal{V}, \mathcal{X}, \dots$, will always denote subsets of M . Let $\langle \mathcal{X} \rangle$ be the submodule spanned by \mathcal{X} . We shall use the letters u, v, x, \dots , for elements of the module M , and $\alpha, \beta, \mu, \dots$, will be elements of the ring R . Using the terminology for abelian p -groups in [3, p.4] we say that $x \in M$ has *exponent* k , and we write $e(x) = k$, if k is the smallest nonnegative integer such that $p^k x = 0$. An element $x \in M$ is said to have (finite) *height* s if $x \in p^s M$ and $x \notin p^{s+1} M$, and x has *infinite height*, if $x \in p^s M$ for all $s \geq 0$. We write $h(x)$ for the height of x . If $x \in W$ then $h_W(x)$ will denote the height of x with respect to W . Note that $e(0) = 0$ and $h(0) = \infty$. Let R^* be the group of units of R . If $\alpha \in R$ is nonzero and $\alpha = p^s \gamma$, $\gamma \in R^*$, then we set $h(\alpha) = s$. We put $h(\alpha) = \infty$ if $\alpha = 0$. We call $x \in M$ an $(s, k; s_1)$ -element if $x \neq 0$ and

$$h(x) = s, e(x) = k, h(p^{k-1}x) = (k-1) + s_1.$$

In accordance with a definition of Baer [1] we say that an element x is *regular* if $h(x) = \infty$ or if $h(x)$ is finite and

$$h(p^j x) = j + h(x), \quad j = 1, \dots, e(x) - 1. \quad (1.2)$$

The two concepts of regularity introduced above are consistent. We shall see in Lemma 3.2 that a finite height element $x \in M$ is regular if and only if $\langle x \rangle$ is a regular submodule of M .

For $s \geq 0, k \geq 0$ we define the submodules $M[p^k] = \{x \in M \mid p^k x = 0\}$ and

$$M_k^s = p^s M \cap M[p^k]. \quad (1.3)$$

Then

$$M_k^s = \{x \in M \mid e(x) \leq k, h(x) \geq s\}.$$

In particular $M_0^s = 0$.

Our main result will be the following.

Theorem 1.1. *Let M be a torsion module over a discrete valuation domain and let W be a submodule of M . The following conditions are equivalent.*

(K) *W is regular, i.e. if $n \geq 0, r \geq 0$ then*

$$p^n W \cap p^{n+r} M = p^n (W \cap p^r M). \quad (1.4)$$

(B) *If $x \in W$ is nonzero then x can be decomposed as*

$$x = y_{k_1}^{s_1} + \cdots + y_{k_m}^{s_m} \quad (1.5)$$

such that

$$y_{k_i}^{s_i} \in W \text{ is regular, } i = 1, \dots, m,$$

and

$$h(y_{k_i}^{s_i}) = s_i, \quad e(y_{k_i}^{s_i}) = k_i,$$

and

$$k_1 > \cdots > k_m > 0 \text{ and } s_1 > \cdots > s_m. \quad (1.6)$$

(FPP) *If $s \geq 0, k \geq 1$, then*

$$(W \cap M_k^{s+1}) + (W \cap M_{k-1}^s) = W \cap (M_k^{s+1} + M_{k-1}^s). \quad (1.7)$$

By a result of Baer [4, p.4, Lemma 65.4] condition (B) is satisfied for $W = M$. Hence (B) singles out those submodules W where each element $x \in W$ allows a decomposition (1.5) such that the summands $y_{k_i}^{s_i}$ can be chosen from W itself. With regard to condition (FPP) we observe that the inclusion

$$(W \cap M_k^{s+1}) + (W \cap M_{k-1}^s) \subseteq W \cap (M_k^{s+1} + M_{k-1}^s) \quad (1.8)$$

holds for all submodules W .

The proof of the theorem will be split into two parts. In Section 3 we show that (B) and (K) are equivalent and in Section 4 we prove the equivalence of (B) and (FPP).

2 Decomposition of elements

We introduce a condition which will be the link between (B) and (K) on one hand and between (B) and (FPP) on the other. For a submodule W we define condition (H) as follows.

(H) If $x \in W$ is an $(s, k; s_1)$ -element then x can be decomposed as

$$x = y_k^{s_1} + z, \quad y_k^{s_1} \in W, \quad z \in W, \quad (2.1)$$

such that

$$h(y_k^{s_1}) = s_1, \quad e(y_k^{s_1}) = k, \quad \text{and} \quad h(z) = s, \quad e(z) < k. \quad (2.2)$$

The following technical lemma will be useful in several instances. It implies that the element $y_k^{s_1}$ in (2.1) is regular.

Lemma 2.1. *Let $x \in M$ be an $(s, k; s_1)$ -element. Assume*

$$x = y + z, \quad z \in M_{k-1}^s. \quad (2.3)$$

Then $y \neq 0$, $e(y) = k$, and

$$s \leq h(y) \leq s_1. \quad (2.4)$$

The element y is regular if and only if $h(y) = s_1$. If x is regular then (2.3) implies $h(y) = s$.

Proof: From (2.3) follows $p^{k-1}y = p^{k-1}x \neq 0$, and $e(y) = k$. Therefore

$$(k-1) + h(y) \leq h(p^{k-1}y) = h(p^{k-1}x) = (k-1) + s_1, \quad (2.5)$$

which yields $h(y) \leq s_1$. It is obvious from (2.5) that we have $h(y) = s_1$ if and only if

$$h(p^{k-1}y) = (k-1) + h(y),$$

i.e., if and only if y is regular. If x is regular then $s_1 = s$ and (2.4) yields $h(y) = s$. ■

Lemma 2.2. *For a submodule W the conditions (B) and (H) are equivalent.*

Proof: There is nothing to prove if x is regular. Thus, in the following we assume that x is a non-regular element of W with $h(x) = s$ and $e(x) = k$. In that case we have $k > 1$, $s_1 > s$, and $h(p^{k-1}x) = (k-1) + s_1$.

(B) \Rightarrow (H) Let x be given as in (1.5), with $m \geq 2$. Put $z = y_{k_2}^{s_2} + \cdots + y_{k_m}^{s_m}$.

Then (1.6) implies $e(z) \leq k_2 < k$ and $h(z) = s_m = s$. Hence the decomposition $x = y_{k_1}^{s_1} + z$ is of type (H) .

(H) \Rightarrow (B) Let x be an $(s, k; s_1)$ -element of W and assume that x is decomposed according to (H) as

$$x = y_k^{s_1} + z \quad (2.6)$$

such that (2.2) holds. We know from Lemma 2.1 that $y_k^{s_1}$ is regular. Consider x with $s_1 > s$, $k > 1$. Assume as an induction hypothesis that condition (H) ensures a decomposition of type (B) for all $w \in W$ with $e(w) < k$. Thus we have

$$z = z_{l_2}^{t_2} + \cdots + z_{l_m}^{t_m}, \quad m \geq 2,$$

with properties in accordance with (B). Thus $h(p^{l_2-1}z) = (l_2-1) + t_2$, $t_2 \geq s$, and $t_2 > \cdots > t_m = s = h(z)$, and $k > e(z) = l_2 > \cdots > l_m > 0$. If $s_1 > t_2$ then we already have the desired decomposition. Now suppose $t_2 \geq s_1$. Let j be such that

$$t_2 > \cdots > t_j \geq s_1 > t_{j+1}. \quad (2.7)$$

Note that $t_m \geq s_1$ can not occur because of $t_m = s$ and $s_1 > s$. Set

$$v = y_k^{s_1} + (z_{l_2}^{t_2} + \cdots + z_{l_j}^{t_j}).$$

Then $k > l_2$ yields $e(v) = k$. Since $y_k^{s_1}$ is regular we see that $p^{k-1}v = p^{k-1}y_k^{s_1}$ implies $(k-1) + s_1 = h(p^{k-1}v)$. Hence $h(v) \leq s_1$. On the other hand it follows from (2.7) that $h(v) \geq s_1$. Therefore $h(v) = s_1$, and v is regular. If we rewrite (2.6) in the form

$$x = v + z_{l_{j+1}}^{t_{j+1}} + \cdots + z_{l_m}^{t_m},$$

then we have a decomposition with $h(v) = s_1$ and $s_1 > t_{j+1} > \cdots > t_m = s$ and $e(v) = k > l_{j+1} > \cdots > l_m > 0$. \blacksquare

It is not difficult to check that the following observation characterizes the numbers m , k_i and s_i in (1.5). For a nonzero element $x \in M$ with $e(x) = k$ define $g(x) = h(x) + e(x)$.

Lemma 2.3. *Let $x \in M$ be decomposed as*

$$x = y_{k_1}^{s_1} + \cdots + y_{k_m}^{s_m} \quad (2.8)$$

such that

$$h(y_{k_i}^{s_i}) = s_i, \quad e(y_{k_i}^{s_i}) = k_i, \quad \text{and } y_{k_i}^{s_i} \text{ is regular, } i = 1, \dots, m.$$

and

$$k_1 > \cdots > k_m > 0 \text{ and } s_1 > \cdots > s_m.$$

Set $K = \{k_1, \dots, k_m\}$. Then $j \in \{1, \dots, k-1\}$, is in K if and only if $g(p^j x) > g(p^{j-1} x)$. Moreover

$$h(p^{k_j-1} x) = (k_j - 1) + s_j, \quad j = 1, 2, \dots, m.$$

In particular, we have $e(x) = k_1$ and $h(x) = s_m$.

3 Equivalence of (K) and (B)

Condition (K) can be reformulated in a more convenient form.

Lemma 3.1. *We have*

$$p^n W \cap p^{n+r} M = p^n (W \cap p^r M), \quad n \geq 0, r \geq 0, \quad (3.1)$$

if and only if for each $w \in W$ with $h(p^n w) = n + r$ there exists an element $\tilde{w} \in W$ such that

$$p^n w = p^n \tilde{w} \text{ and } h(\tilde{w}) = r. \quad (3.2)$$

Proof: Obviously (3.1) is equivalent to

$$p^n W \cap p^{n+r} M \subseteq p^n (W \cap p^r M), \quad n \geq 0, r \geq 0. \quad (3.3)$$

Now (3.3) holds if and only if

$$x \in p^n W, \quad x \in p^{n+r} M \text{ and } x \notin p^{n+r+1} M$$

imply $x \in p^n (W \cap p^r M)$. That implication means the following. If $x = p^n w$ and $w \in W$ and $h(x) = n + r$, then $x = p^n \tilde{w}$ for some $\tilde{w} \in W$ with $h(\tilde{w}) \geq r$. Because of $h(p^n \tilde{w}) = n + r$ the inequality $h(\tilde{w}) \geq r$ is equivalent to $h(\tilde{w}) = r$. ■

Lemma 3.2. *Let x be an element of finite height with $e(x) = k$. Then x is regular if and only if the submodule $\langle x \rangle$ is regular, i.e.*

$$p^r \langle x \rangle \cap p^{n+r} M = p^r (\langle x \rangle \cap p^n M), \quad n \geq 0, r \geq 0. \quad (3.4)$$

Proof: Assume (3.4). We want to show that $h(p^{k-1}x) = (k-1) + s_1$ implies $s_1 = h(x)$. According to Lemma 3.1 there exists an element $\tilde{x} \in \langle x \rangle$ with properties corresponding to (3.2), i.e. $\tilde{x} = \gamma p^t x$, $\gamma \in R^*$, and $p^{k-1}x = p^{k-1}(\gamma p^t x)$ and $h(p^t x) = s_1$. Then we have $t = 0$, and $h(x) = s_1$. It is easy to check that (3.4) holds if x is regular. ■

Proof of Theorem 1.1, Part I: (B) \Leftrightarrow (K)

(B) \Rightarrow (K) We want to show that condition (B) implies (K) in the equivalent form of Lemma 3.1. Let $w \in W$ be such that $h(p^n w) = n + r$, and $h(w) = s$, $e(w) = k_1$. Then $s \leq r$ and $k_1 > n$. Hence (B) yields a decomposition

$$w = y_{k_1}^{s_1} + \cdots + y_{k_m}^{s_m}$$

where the elements $y_i^{s_i} \in W$ are regular, $h(y_i^{s_i}) = s_i$, and

$$s_1 > \cdots > s_m = s = h(w)$$

and $e(w) = k_1 > \cdots > k_m > 0$. Let t be such that $k_t > n \geq k_{t+1}$. Then

$$n + r = h(p^n w) = h(p^n y_{k_1}^{s_1} + \cdots + p^n y_{k_t}^{s_t}),$$

and $h(p^n w) = h(p^n y_{k_t}^{s_t}) = n + s_t$. Hence $s_t = r$. Set $\tilde{w} = y_{k_1}^{s_1} + \cdots + y_{k_t}^{s_t}$. Then $\tilde{w} \in W$ and $h(\tilde{w}) = r$ and $p^n w = p^n \tilde{w}$.

(K) \Rightarrow (B) Because of Lemma 2.2 it suffices to show that (K) implies (H). Let $x \in W$ be an $(s, k; s_1)$ -element. Set $w = p^{k-1}x$. Then (K), resp. Lemma 3.1, imply that there exists an $\tilde{x} \in W$ such that

$$p^{k-1}x = p^{k-1}\tilde{x} \tag{3.5}$$

and $h(\tilde{x}) = s_1$. From (3.5) follows $e(\tilde{x}) = k$ and $h(p^{k-1}\tilde{x}) = (k-1) + s_1$. Now set $z = x - \tilde{x}$. Then (3.5) yields $e(z) < k$. Hence $x = \tilde{x} + z$ is a decomposition of type (H). ■

As (K) holds for $W = M$ we can write each nonzero element x of M according to (H) in the form (2.1). Similarly we can decompose x according to (B) as a sum of the form (1.5). In that case we recover the result of Baer [4, p.4, Lemma 65.4] mentioned in Section 1.

4 Equivalence of (B) and (FPP)

In [2] J. Ferrer, and F. and X. Puerta studied marked invariant subspaces of an endomorphism A of \mathbb{C}^n . Their investigation is based on subspaces of the

form $\text{Im}(\lambda I - A)^s \cap \text{Ker}(\lambda I - A)^k$. Thus the submodules M_k^s in (1.3) are a generalization of those subspaces. The next lemma is adapted from [2]. It characterizes regular elements in terms of M_k^s . Note that $M_k^s \subseteq M_{k_1}^{s_1}$ if $s_1 \leq s$ and $k \leq k_1$. Hence $M_k^{s+1} + M_{k-1}^s \subseteq M_k^s$.

Lemma 4.1. *An element $x \in M$ satisfies*

$$x \in M_k^s \text{ and } x \notin M_k^{s+1} + M_{k-1}^s \quad (4.1)$$

if and only if

$$x \text{ is regular and } h(x) = s \text{ and } e(x) = k. \quad (4.2)$$

Proof: “ \Rightarrow ” Assume that x satisfies (4.1). Recall that $x \in M_k^s$ if and only if both $h(s) \geq s$ and $e(x) \leq k$. Hence

$$x \notin M_k^{s+1} + M_{k-1}^s \quad (4.3)$$

implies $h(x) = s$ and $e(x) = k$. Assume $h(p^{k-1}x) = (k-1) + s_1$. If we decompose x according to (H) then $x = y_k^{s_1} + z \in M_k^{s_1} + M_{k-1}^s$. Hence (4.3) implies $s_1 = s$, and x is regular.

“ \Leftarrow ” Consider an element x with properties (4.2). Then $x \in M_k^s$, $x \notin M_k^{s+1}$ and $x \notin M_{k-1}^s$. If

$$x = y + z, \quad y \neq 0, \quad z \in M_{k-1}^s,$$

and x is regular, then it follows from Lemma 2.1 that $h(y) = s$. Hence we have $y \notin M_k^{s+1}$ and $x \notin M_k^{s+1} + M_{k-1}^s$. ■

Proof of Theorem 1.1, Part II: (H) \Leftrightarrow (FPP)

(H) \Rightarrow (FPP) Because of the inclusion (1.8) the identity (1.7) in (FPP) is equivalent to

$$W \cap (M_k^{s+1} + M_{k-1}^s) \subseteq (W \cap M_k^{s+1}) + (W \cap M_{k-1}^s). \quad (4.4)$$

We want to show that condition (H) implies (4.4) for all $s \geq 0$, $k \leq 1$. Take an element

$$x \in W \cap (M_k^{s+1} + M_{k-1}^s). \quad (4.5)$$

Then $x \in M_k^s$ and therefore $h(x) \geq s$ and $e(x) \leq k$. To prove that

$$x \in (W \cap M_k^{s+1}) + (W \cap M_{k-1}^s) \quad (4.6)$$

we consider three cases. First, let $h(x) \geq s+1$ then $x \in W \cap M_k^{s+1}$ and (4.6) is obvious. Secondly, let $e(x) \leq k-1$. In that case $x \in W \cap M_{k-1}^s$. Now assume $h(x) = s$ and $e(x) = k$. By Lemma 4.1 it follows from (4.5) that x is

not regular. Hence $h(p^{k-1}x) = (k-1) + s_1$ and $s_1 > s$. According to (H) we have $x = y_k^{s_1} + z$ with $y_k^{s_1} \in W \cap M_k^{s_1}$ and $z \in W \cap M_{k-1}^s$, which yields (4.6).

(FPP) \Rightarrow (H) Let x be an $(s, k; s_1)$ -element. If $s_1 = s$ then x is regular and we have (2.1) with $z = 0$. Suppose now that x is not regular, i.e. $s_1 \geq s+1$. Then Lemma 4.1 implies $x \in W \cap (M_k^{s+1} + M_{k-1}^s)$. From (FPP) we obtain

$$x = y + z, \quad y \in W \cap M_k^{s+1}, \quad z \in W \cap M_{k-1}^s. \quad (4.7)$$

Then $y \neq 0$, $e(y) = k$ and $h(y) \geq s+1$. Let y in (4.7) be such that $h(y)$ is maximal. We shall see that such a choice of y implies $h(y) = s_1$, and in that case (4.7) is a decomposition of type (H). Now suppose that $h(y) = \tilde{s} < s_1$. Then, by Lemma 2.1, the element $y \in W$ is not regular. Applying Lemma 4.1 to $y \in W \cap M_k^{\tilde{s}}$ we obtain $y \in W \cap (M_k^{\tilde{s}+1} + M_{k-1}^{\tilde{s}})$. Thus (FPP) yields

$$y = \tilde{y} + z_2, \quad \tilde{y} \in W \cap M_k^{\tilde{s}+1}, \quad \tilde{y} \neq 0, \quad z_2 \in W \cap M_{k-1}^{\tilde{s}}.$$

Hence $x = \tilde{y} + (z + z_2)$, and we have another decomposition of the form (4.7), but now with $h(\tilde{y}) > \tilde{s}$, which contradicts the maximality of \tilde{s} . \blacksquare

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